

Without any further ado, I present (at least one set of) the Dirac γ matrices:

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{w/ } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $\gamma^4 = -i \begin{pmatrix} \sigma_0 & \sigma_0 & 0 & -i \\ \sigma_0 & \sigma_0 & i & 0 \\ 0 & i & \sigma_0 & 0 \\ -i & 0 & \sigma_0 & \sigma_0 \end{pmatrix}$

These have some nice properties:

- Recall $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4 \times 4}$
- Then $(\gamma^0)^2 = -1, (\gamma^i)^2 = 1$
- And $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0$ if $\mu \neq \nu$ since $\eta^{\mu\nu}$ is diagonal
or $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$

We can now explicitly form the generators:

$$\sigma^{0i} = -\frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{4} [\gamma^0 \gamma^i - \gamma^i \gamma^0]$$

$$= -\frac{i}{4} \left[-i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-i) \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} (-i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

Note: We now see why we needed the $\frac{i}{4}$ in the definition. The transformation now reduces to the usual $SU(2)$ transformation on each pair of spinor indices, i.e. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \} SU(2)$
 $\psi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \} SU(2)$

$$\sigma^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad \text{e.g. } \sigma^{12} = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Last time we tried to build an invariant $\tilde{\psi}^t \psi$ from spinors trying $\tilde{\psi} = \psi$ (like we did with $SU(2)$),

but... $\tilde{\psi}^t \psi = \psi^t \psi \rightarrow (\psi')^t \psi' = (e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi)^t e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi$

Recall: $\sigma^{0i} = \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$ $\sigma^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$
 and $\sigma^{\mu\nu t} = \sigma^{\mu\nu}$

$$= \psi^t \underbrace{(e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}})^t}_{\text{But the } \sigma^{\mu\nu} \text{ are not all Hermitian!}} e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi$$

But the $\sigma^{\mu\nu}$ are not all Hermitian!

In particular $\sigma^{0i t} = -\sigma^{0i}$ Not-Hermitian
 $\sigma^{ij t} = \sigma^{ij}$ Hermitian

The reason $\psi^t \psi$ worked for $SU(2)$ is that the generators were all Hermitian.

But we can fix this with a different choice of dual: $\tilde{\psi} = i\gamma^0 \psi$

Then: $\tilde{\psi}^t \psi = (i\gamma^0 \psi)^t \psi$

$$\uparrow \gamma^0 = -i \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

$$= -i \psi^t \gamma^{0t} \psi$$

$$= i \psi^t \gamma^0 \psi \rightarrow i (\psi')^t \gamma^0 \psi' = i (e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi)^t \gamma^0 e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi$$

$$= i \psi^t \underbrace{(e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}})^t}_{\gamma^0 e^{-\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}}} \gamma^0 e^{\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}} \psi$$

You will show this in your HW

Same! $\rightarrow = i \psi^t \gamma^0 \psi$

So in the end we define our dual spinor with $\tilde{\psi} = i\gamma^0 \psi$ and the adjoint $\bar{\psi} \equiv \tilde{\psi}^t = i\psi^t \gamma^0$

Or in other words $g = i\gamma^0 \Rightarrow \tilde{\psi} = g\psi$

\Downarrow
 $\bar{\psi} \psi$ is invariant

Now that we have an invariant we can include spinor degrees of freedom in our invariant Lagrangian.

But we can do more! The γ^m matrices have the unique property of "linking" spin space to spacetime. Recall they secretly have 2 spin indices which we omit γ_{ab} (that's what makes them matrices) and they obviously have one spacetime index. We might wonder if this truly makes the γ^m a vector in spacetime. More on that and what we can do with it in a moment.

Technical aside: Spin space is tied to the geometry at hand. In fact just as vectors actually live in the tangent space at each point, or tangent bundle over all of spacetime, spinors live in the spin bundle over spacetime.

Since both spinors and vectors are tied at some level to spacetime, if we transform coordinates both get impacted whenever such indices appear.

To that end we introduce some abbreviated notation: If: $V^m \rightarrow V^{m'} = \Lambda^{m'}_m V^m$
 $\psi \rightarrow \psi' = e^{\frac{i}{2} \theta^{\alpha\beta} \gamma_{\alpha\beta}} \psi$
 $\xrightarrow{\hspace{10em}} S[\Lambda]$
 The spinor version of Λ

Back to γ^m as a vector. This is one place where we will restore spinor indices. If we say ψ^a transforms as $\psi^a \rightarrow \psi^{a'} = S[\Lambda]^{a'}_a \psi^a$ and $\bar{\psi}_b \rightarrow \bar{\psi}_{b'} = S[\Lambda]^{b'}_b \bar{\psi}_b$.

But for $\bar{\psi}_a \psi^a$ to be invariant we need $\bar{\psi}_a \psi^a \rightarrow \bar{\psi}_{a'} \psi^{a'} = \bar{\psi}_a \underbrace{S[\Lambda]^{a'}_a S[\Lambda]^{a'}_b}_{=I^{a'}_b} \psi^b$

Now the correct way to label γ^m is $\gamma^{m a}_b$. This is so that $\bar{\psi}_a \gamma^{m a}_b \psi^b$ makes sense w/ the Einstein summation convention. If we call $S[\Lambda]^{a'}_a = S[\Lambda]$ then $S[\Lambda]^{a'}_a = S[\Lambda]^{-1}$

To transform $\gamma^{m a}_b$ we simply transform every index in sight:

$$\gamma^{m a}_b \rightarrow \gamma^{m' a'}_{b'} = \Lambda^{m'}_m S[\Lambda]^{a'}_a S[\Lambda]^{b'}_b \gamma^{m a}_b = \Lambda^{m'}_m S[\Lambda]^{a'}_a \gamma^{m a}_b S[\Lambda]^{b'}_b$$

Now hiding the spinor indices this becomes: $\gamma^m \rightarrow \gamma^{m'} = \Lambda^{m'}_m S[\Lambda] \gamma^m S[\Lambda]^{-1}$

In practice we usually only use the vector index to transform. The reason is that whenever a γ^m appears in say a Lagrangian, it is in the middle of a "spinor sandwich", e.g. $\bar{\psi} \gamma^m \psi$.

The need for spinor sandwiches should be somewhat apparent... they have no overall spin indices, so are invariant spin-wise (but may still have vector indices to take care of!).

In fact: $\bar{\psi} \psi$ - scalar
 $\bar{\psi} \gamma^m \psi$ - vector
 $\bar{\psi} \gamma^m \gamma^{\nu} \psi$ - (2,0) tensor
 etc.

You can think of spinor sandwiches in terms of matrix multiplication of spin indices:

$$\bar{\psi} \psi \Rightarrow (\dots) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\cdot)$$

$$\bar{\psi} \gamma^m \psi \Rightarrow (\dots) \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = (\cdot)$$

↳ or insert as many matrices as you like!

$$\bar{\psi} \gamma^m \psi \rightarrow \bar{\psi}' \gamma^{m'} \psi' = \bar{\psi} S[\Lambda]^{-1} S[\Lambda] \Lambda^{m'}_m \gamma^m S[\Lambda]^{-1} S[\Lambda] \psi$$

$$= \Lambda^{m'}_m \underbrace{\bar{\psi} \gamma^m \psi}_{\text{transforms like a vector}}$$

Actions

Of the various formalisms for describing mechanics, the Lagrangian and associated action principle is the most useful for our purposes since it can be generalized in several ways:

$$\begin{array}{l} \underline{NR \rightarrow \text{Relativistic}}, \quad \underline{PP \rightarrow \text{Fields}}, \quad \underline{CM \rightarrow QM} \\ \int dt \rightarrow \int d^4x \quad x(z) \rightarrow \phi(x^\mu) \quad \delta S = 0 \rightarrow \int D\phi e^{i\hbar^{-1}S} \end{array}$$

and because it allows us to work with manifest symmetries.

Preliminary:

The action is a functional which essentially means a function of functions.

For example, whereas a function takes an argument and returns a number: $f(x) = x^2 \Rightarrow f(2) = 4$
a functional does the same when we insert a function: $S[f(x)] = \int_0^1 f(x) dx$

$$\Downarrow \\ S[x^2] = \int_0^1 x^2 dx = \frac{1}{3}$$

What makes this nice is that in the same way we can extremize a function w/ $\frac{df}{dx} = 0$ to find the value of x which achieves the max or min of $f(x)$, we can also extremize a functional w/ $\frac{\delta S}{\delta f} = 0$ to find the function $f(x)$ which achieves the max or min of $S[f(x)]$.

You should be familiar w/ $S = \int L dt$ $L(q, \dot{q}) = T - V$ and $\delta S = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$

We need a relativistic version of this suitable for quantum use.

Euler-Lagrange e.o.m.